A general theory of minimal increments for Hirsch-type indices and applications to the mathematical characterization of Kosmulski-indices

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ABSTRACT

For a general function $f(n)$ $(n=1,2,...)$, defining general Hirsch-type indices, we can characterize the first increment $I_1(n)=(n+1)f(n+1)-nf(n)$ as well as the second increment $I_2(n)=I_1(n+1)-I_1(n+1)$. An application is given by presenting mathematical characterizations of Kosmulski-indices.

Keywords: Increment; Hirsch-type index;Kosmulski-index.

INTRODUCTION

Let us have a set of papers where the ith paper has c_i citations. We assume that papers are arranged in decreasing order of received citations (i.e. $c_i \geq c_j$ if and only if $i \leq j$). The most general Hirsch-type index can be defined as follows. Let $f(n)$ $(n=1,2,3,...)$ be a general increasing function. Then the Hirsch-type index (based on *f*) for this set of papers and citations is the highest rank *n* such that all the papers on ranks $1,...,n$ have at least $f(n)$ citations. Examples are $f(n)=n$ for the Hirsch-index (Hirsch (2005)), $f(n)=an$ $(a>0)$ for the general Wu-index (Egghe (2011) and Wu (2010)for $a=10$) and $f(n)=n^a$ $(a>0)$ for the general Kosmulski-index (Egghe (2011) and Kosmulski (2006) for $a=2$). Note that the general Wu- and Kosmulski-indices reduce to the Hirsch-index for $a=1$.

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Given such a function $f(n)$ $(n=1,2,3,...)$ the minimum situation to have an index equal to *n* is to have *n* papers with exactly $f(n)$ citations each and where the other papers have zero citations. In this case we have a total of $nf(n)$ citations. The minimum situation to have an index equal to $n+1$ is $n+1$ papers each having $f(n+1)$ citations and where the other papers have zero citations. Here we have $(n+1)f(n+1)$ citations in total, hence an increase of $(n+1)f(n+1)-nf(n)$ citations. We define (see also Egghe (2013 a,b)) the general increment of order 1 as, for every $n=1,2,3,...$

$$I_1(n)=(n+1)f(n+1)-nf(n) \tag{1}$$

The general increment of order 2 is defined as

$$I_2(n)=I_1(n+1)-I_1(n) \tag{2}$$

which is equal to, by (1)

$$I_2(n)=(n+2)f(n+2)-2(n+1)f(n+1)+nf(n) \tag{3}$$

for all $n=1,2,3,...$

Examples (see also egghe (2013 a, b)):

1. For the general Wu-index ($f(n)=an$) we have

$$I_1(n)=a(2n+1) \tag{4}$$

$$I_2(n)=2a \tag{5}$$

   for all *n*. This gives for the Hirsch-index $(a=1)$:

$$I_1(n)=2n+1 \tag{6}$$

$$I_2(n)=2 \tag{7}$$

   for all *n*.

2. For the general Kosmulski-index ($f(n)=a^n$) we have

$$I_1(n)=(n+1)^{a+1}-n^{a+1} \tag{8}$$

$$I_2(n)=(n+2)^{a+1}-2(n+1)^{a+1}+n^{a+1} \tag{9}$$

   for all *n*.

3. For the threshold index ($f(n) = C$, a constant) we have

$$I_1(n) = C \tag{10}$$

$$I_2(n) = 0 \tag{11}$$

for all *n*.

In Egghe (2013 a, b) we characterized the general Wu-index (hence also the Hirsch-index) and the threshold index using the increments $I_1(n)$ and $I_2(n)$. In the present paper we will characterize general Hirsch-type-indices (given a function $f(n)$) by means of their first and second increments given a certain function $I_1(n) = \varphi(n)$ and $I_2(n) = \psi(n)$. This will be done in the next section. In the third section we apply this general theory to the characterization of the Kosmulski-indices by means of the increments (8) and (9). The paper closes with conclusions and suggestions for further research.

CHARACTERIZATIONS OF GENERAL HIRSCH-TYPE-INDICES USING THE INCREMENTS $I_1(n)$ AND $I_2(n)$

Let us have a general Hirsch-type-index defined by a general increasing function $f(n)$.
For a general function $\varphi(n)$ we have:

Theorem 1: The following assertions are equivalent:

(i) $I_1(n) = \varphi(n)$ (12)
for all $n = 1, 2, 3, \ldots$

(ii) $f(n) = \frac{f(1)}{n} + \frac{\sum_{i=1}^{n-1} \varphi(i)}{n}$ (13)
for all $n = 1, 2, 3, \ldots$

Proof: (i) => (ii)

Since $I_1(n) = \varphi(n)$, for all *n*, we have by (1)

$$(n+1) f(n+1) - nf(n) = \varphi(n) \tag{14}$$

Hence, from (14),

$$f(n+1) = \frac{n}{n+1} f(n) + \frac{\varphi(n)}{n+1} \tag{15}$$

Choosing the free parameter $f(1)>0$ we hence have from (15)

$$f(2)=\frac{1}{2}f(1)+\frac{\varphi(1)}{2} \tag{16}$$

$$f(3)=\frac{1}{3}f(1)+\frac{\varphi(1)+\varphi(2)}{3} \tag{17}$$

(using also (16)). So we have shown that (13) is true for $n=1,2,3$. Complete induction supposes (13) to be true for *n* and we have to show (13) for *n* replaced by $n+1$: By (15) and (13) (for *n*)

$$f(n+1)=\frac{n}{n+1}\left[\frac{f(1)}{n}+\frac{\sum_{i=1}^{n-1}\varphi(i)}{n}\right]+\frac{\varphi(n)}{n+1}$$

$$f(n+1)=\frac{1}{n+1}f(1)+\frac{\sum_{i=1}^{n}\varphi(i)}{n+1}$$

which is (13) for *n* replaced by $n+1$.

(ii) => (i)

By (13) (applied to *n* and $n+1$) we have

$$\begin{aligned} I_1(n) &= (n+1)f(n+1)-nf(n) \\ &= (n+1)\left[\frac{1}{n+1}f(1)+\frac{\sum_{i=1}^{n}\varphi(i)}{n+1}\right]-n\left[\frac{1}{n}f(1)+\frac{\sum_{i=1}^{n-1}\varphi(i)}{n}\right] \end{aligned}$$

Hence $I_1(n)=\varphi(n)$ as is readily seen. Hence we proved (i), completing the proof of this theorem.

Using the second increment yields another characterization of general Hirsch-type-indices. For a general function $\psi(n)$ we have:

<u>Theorem 2:</u> The following assertions are equivalent:

(i) $I_2(n)=\psi(n)$ (18)

for all $n=1,2,3,...$

(ii) $$f(n)=\frac{1}{n}\left[2(n-1)f(2)-(n-2)f(1)+\sum_{i=1}^{n-2}(n-i-1)\psi(i)\right] \quad (19)$$

for all $n=1,2,3,\ldots$

<u>Proof:</u> (i) => (ii)

From (18) and (3) we have

$$I_2(n)=(n+2)f(n+2)-2(n+1)f(n+1)+nf(n)=\psi(n) \quad (20)$$

for all *n*. Hence

$$f(n+2)=\frac{2(n+1)}{n+2}f(n+1)-\frac{n}{n+2}f(n)+\frac{\psi(n)}{n+2} \quad (21)$$

So we choose two free parameters $f(2)\geq f(1)>0$ (to obtain an increasing function $f(n)$) and this gives, using (21):

$$f(3)=\frac{4}{3}f(2)-\frac{1}{3}f(1)+\frac{\psi(1)}{3} \quad (22)$$

$$f(4)=\frac{6}{4}f(2)-\frac{2}{4}f(1)+\frac{2}{4}\psi(1)+\frac{\psi(2)}{4} \quad (23)$$

using also (22). So we have that (19) is true for $n=1,2,3,4$ (defining $\sum_{i=1}^{-1}=\sum_{i=1}^{0}=0$).

Complete induction supposes (19) to be true for *n* and $n+1$ and we have to prove (19) for $n+2$. By (21) we have:

$$\begin{aligned}
f(n+2)&=\frac{2(n+1)}{n+2}\frac{1}{n+1}\left[2nf(2)-(n-1)f(1)+\sum_{i=1}^{n-1}(n-1)\psi(i)\right]\\
&-\frac{n}{n+2}\frac{1}{n}\left[2(n-1)f(2)-(n-2)f(1)+\sum_{i=1}^{n-2}(n-i-1)\psi(i)\right]+\frac{\psi(n)}{n+2}\\
&=\frac{1}{n+2}\left[4f(2)-2(n-1)f(1)+2\sum_{i=1}^{n-1}(n-1)\psi(i)-2(n-1)f(2)+(n-2)f(1)-\sum_{i=1}^{n-2}(n-i-1)\psi(i)+\psi(n)\right]\\
&=\frac{1}{n+2}\left[2(n+1)f(2)-nf(1)+2\psi(n-1)+2\sum_{i=1}^{n-2}(n-i)\psi(i)-\sum_{i=1}^{n-2}(n-i-1)\psi(i)+\psi(n)\right]\\
&=\frac{1}{n+2}\left[2(n+1)f(2)-nf(1)+\sum_{i=1}^{n}(n-i+1)\psi(i)\right]
\end{aligned}$$

which is (19) with *n* replaced by $n+2$.

(ii) => (i)

By (19), applied to *n*, $n+1$ and $n+2$ we have, by (3):

$$\begin{aligned}
I_2(n) &= \frac{n+2}{n+2}\left(2(n+1)f(2)-nf(1)+\sum_{i=1}^{n}(n-i-1)\psi(i)\right)\\
&-\frac{2(n+1)}{n+1}\left(2nf(2)-(n-1)f(1)+\sum_{i=1}^{n-1}(n-1)\psi(i)\right)\\
&+\frac{n}{n}\left(2(n-1)f(2)-(n-2)f(1)+\sum_{i=1}^{n-2}(n-i-1)\psi(i)\right)\\
&=\psi(n)
\end{aligned}$$

as is readily seen. This completes the proof of (ii) => (i) and hence the proof of theorem.

Based on this theory we will, in the next section, give two characterizations of the general Kosmulski-indices.

CHARACTERIZATIONS OF THE GENERAL KOSMULSKI-INDICES

For $I_1(n)$ as in (8) we have the following theorem.

Theorem 3: The following assertions are equivalent:

(i) $$I_1(n)=(n+1)^{a+1}-n^{a+1} \qquad (24)$$

for all $n=1,2,3,...$

(ii) $$f(n)=\frac{f(1)}{n}+\frac{1}{n}\left(n^{a+1}-1\right) \qquad (25)$$

Proof: This follows from Theorem 1 and the fact that

$$\begin{aligned}
\sum_{i=1}^{n-1}\psi(i) &= n^{a+1}-(n-1)^{a+1}+(n-1)^{a+1}\\
&-(n-2)^{a+1}+...-2^{a+1}-1\\
&=n^{a+1}-1
\end{aligned}$$

The next corollary gives a characterization of the general Kosmulski-indices using the first increment.

<u>Corollary 4:</u> The following assertions are equivalent

(i) $f(1)=1$ and

$$I_1(n)=(n+1)^{a+1}-n^{a+1}$$

for all $n=1,2,3,...$

(ii) $f(n)=n^a$

for all $n=1,2,3,...$, hence the general Kosmulski-indices.

<u>Proof:</u> This follows readily from theorem 3.

For $I_2(n)$ as in (9) we have the following theorem:

<u>Theorem 5:</u> The following assertions are equivalent:

(i) $I_2(n)=(n+2)^{a+1}-2(n+1)^{a+1}+n^{a+1}$ (26)

for all $n=1,2,3,...$

(ii)

$$f(n)=\frac{1}{n}[2(n-1)f(2)-(n-2)f(1) \\ +(n-2)\left(-2.2^{a+1}+1\right)+(n-3)2^{a+1}+n^{a+1}] \quad (27)$$

<u>Proof:</u> (19) transforms into

$$f(n)=\frac{1}{n}\left[2(n-1)f(2)-(n-2)f(1)+\sum_{i=1}^{n-2}(n-i-1)\left[(i+2)^{a+1}-2(i+1)^{a+1}+i^{a+1}\right]\right]$$

whichis (27) since all coefficients of $3^{a+1},4^{a+1},...,(n-1)^{a+1}$ are zero which is readily seen by evaluation of all the Σs. So Theorem 5 follows from Theorem 2.

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The next corollary gives a characterization of the general Kosmulski-indices using the second increment.

<u>Corollary 6:</u> The following assertions are equivalent:

(i) $f(1)=1$, $f(2)=2^a$ and $I_2(n)=(n+2)^{a+1}-2(n+1)^{a+1}+n^{a+1}$
for all $n=1,2,3,...$

(ii) $f(n) = n^a$

for all $n = 1, 2, 3, \ldots$, hence the general Kosmulski-indices.

<u>Proof:</u> This follows readily from Theorem 5 and the fact that

$$f(n) = \frac{1}{n}\left[2(n-1)2^a - (n-2) + (n-2)\left(-2.2^{a+1}+1\right) + (n-3)2^{a+1} + n^{a+1}\right]$$
$$= \frac{1}{n}\left[2^{a+1}\left((n-1) - 2(n-2) + n - 3\right) - (n-2) + (n-2) + n^{a+1}\right]$$
$$= n^a$$

Remark

It is clear from Theorems 3 and 5 and Corollaries 4 and 6 that the incremental identities (24) and (26) yield impact measures that are a generalization of the Kosmulski-indices due to the fact that they contain a parameter $f(1)$ and parameters $f(1)$ and $f(2)$ respectively.

This is a remarkable fact but is in line with the results obtained in Egghe (2013 a,b) on the Wu- and Hirsch-indices.

CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

We have characterized general Hirsch-type indices by means of their first and second increments. They indicate what effort is necessary to increase such an impact measure from *n*to $n+1$, for every $n = 1, 2, 3, \ldots$.

We also characterized the general Kosmulski-indices by means of their increments of first and second order.

Since we treated the most general Hirsch-type indices, this finishes this type of study but leaves open the similar treatment of impact measures which are not of Hirsch-type such as e.g. the impact factor. In this context, impact measures such as the g-index (Egghe (2006)) or the R-index (Jin et al. (2007)) fall in the category of h-type indices since, by using increments $I_1(n)$ and $I_2(n)$, we only consider papers with an equal number $f(n)$ of citations.

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REFERENCES

Egghe, L. 2006. Theory and practice of the g-index. *Scientometrics,* Vol.69, no. 1: 131-152.

Egghe, L. 2011. Characterizations of the generalized Wu- and Kosmulski-indices in Lotkaian systems. *Journal of Informetrics*, Vol. 5, no. 3: 439-445.

Egghe, L. 2013a. A mathematical characterization of the Hirsch-index by means of minimal increments. *Journal of Informetrics,* Vol. 7, no. 2: 388-393.

Egghe, L. 2013b. Mathematical characterizations of the Wu- and Hirsch-indices using two types of minimal increments. Proceedings of the *14th International Society of Scientometrics and Informetrics Conference,* Vienna (Austria), J. Gorraiz, E. Schiebel, C. Gumpenberger, M. Hörlesberger and H. Moed (eds.), 1159-1169, Austrian Institute of Technology.

Hirsch, J.E. 2005. An index to quantify an individual's scientific research output. *Proceedings of the National Academy of Sciences of the United States of America,* Vol. 102, no. 46: 16569-16572.

Jin, B., Liang, L., Rousseau, R. and Egghe, L. 2007. The R- and AR-indices: Complementing the h-index. *Chinese Science Bulletin,* Vol. 52, no. 6: 855-863.

Kosmulski, M. 2006. A new Hirsch-type index saves time and works equally well as the original h-index. *ISSI Newsletter*, Vol. 2, no. 3: 4-6.

Wu, Q. 2010. The w-index: A measure to assess scientific impact by focusing on widely cited papers. *Journal of the American Society for Information Science and Technology*, Vol. 61, no. 3: 609-614.